

How tetraquarks can generate a second chiral phase transition

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Abstract

We consider how tetraquarks can affect the chiral phase transition in theories like QCD, with light quarks coupled to three colors. For two flavors the tetraquark field is an isosinglet, and its effect is minimal. For three flavors, however, the tetraquark field transforms in the same representation of the chiral symmetry group as the usual chiral order parameter, and so for very light quarks there *may* be two chiral phase transitions, which are both of first order. In QCD, results from the lattice indicate that any transition from the tetraquark condensate is a smooth crossover. In the plane of temperature and quark chemical potential, though, a crossover line for the tetraquark condensate is naturally related to the transition line for color superconductivity. For four flavors we suggest that a triquark field, antisymmetric in both flavor and color, combine to form hexaquarks.

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I. INTRODUCTION

As suggested first by Jaffe [1], it is most plausible that in QCD, the lightest scalar mesons with $J^P = 0^+$ are composed not just of a quark and anti-quark, but contain a significant admixture of tetraquark states, with two quarks and two anti-quarks [2–10]. Recently, there is increasing experimental evidence for tetraquark and even pentaquark states of heavy quarks [11].

In this paper we concentrate on light quarks, and generalize the standard analysis of the chiral phase transition at nonzero temperature [12–15] to consider how tetraquarks can affect the chiral transition [16, 17]. We limit ourselves to three colors, and start with the case of two flavors, showing that tetraquarks probably have a small effect on the chiral phase transition. For three flavors, though, if the quarks are sufficiently light then it is possible — although not guaranteed — that the tetraquark field generates a *second* chiral phase transition. In the chiral limit both chiral phase transitions are of first order. We discuss implications for the phase diagram of QCD at nonzero temperature and chemical potential, and conclude with some speculations about four flavors.

A detailed comparison of models with tetraquarks to the hadronic spectrum is necessarily complicated, and involves not just the masses of hadronic states, but their decays [1–10]. Thus our discussion is largely qualitative, to emphasize what we find is an unexpected relation between hadronic phenomenology at zero temperature and the phase transitions of QCD.

II. NOTATION

Left and right handed quarks and anti-quarks are defined as

$$q_{L,R} = \mathcal{P}_{L,R} q \ ; \ \bar{q}_{L,R} = \bar{q} \mathcal{P}_{R,L} \ ; \ \mathcal{P}_{L,R} = \frac{1 \pm \gamma_5}{2} \ , \quad (1)$$

with $\gamma_5^2 = 1$.

We assume there are N_f flavors of massless quarks, which transform under the chiral symmetry group of $SU(N_f)_L \times SU(N_f)_R \times U(1)_A$ as

$$q_L \rightarrow e^{-i\alpha/2} U_L q_L \ ; \ \bar{q}_L \rightarrow e^{+i\alpha/2} \bar{q}_L U_L^\dagger \ ; \ q_R \rightarrow e^{+i\alpha/2} U_R q_R \ ; \ \bar{q}_R \rightarrow e^{-i\alpha/2} \bar{q}_R U_R^\dagger \ ; \quad (2)$$

$U_{L,R}$ are elements of $SU(N_f)_{L,R}$ and α is an axial rotation in $U(1)_A$.

For most of our discussion we implicitly limit ourselves to the case of nonzero temperature and zero quark chemical potential. This allows us to assume that the $U(1)$ symmetry for quark number remains unbroken. At nonzero chemical potential color superconductivity can occur, which spontaneously breaks this $U(1)$ symmetry [18–20]. As discussed in Sec. (VII), the generalization to nonzero quark chemical potential requires a separate analysis.

To construct the effective fields it helps to explicitly denote the flavor and color indices. The quark field q^{aA} , where $a = 1 \dots N_f$ is the flavor index for N_f flavors, and $A = 1 \dots N_c$ for N_c colors. The usual order parameter for chiral symmetry is given by combining a left handed anti-quark and a right handed quark as a color singlet,

$$\Phi^{ab} = \bar{q}_L^{bA} q_R^{aA} . \quad (3)$$

This field transforms as $\bar{\mathbf{N}}_f \times \mathbf{N}_f$ under $SU(N_f)_L$ and $SU(N_f)_R$:

$$\Phi \rightarrow e^{+i\alpha} U_R \Phi U_L^\dagger . \quad (4)$$

Under the axial $U(1)_A$ symmetry we can choose the convention that Φ has charge $= +1$.

We note that the combination of anti-quark and a quark with the same chirality automatically vanishes: *e.g.*, $\bar{q}_L q_L = \bar{q} \mathcal{P}_R \mathcal{P}_L q = 0$. In contrast, for tetraquarks it is possible to pair two diquark fields of the same chirality, Eq. (9) and Sec. (IV B)

The chirally invariant couplings of quarks to the gauge field A_μ and to the chiral field Φ are

$$\mathcal{L}_\Phi^{qk} = \bar{q}_L \not{D} q_L + \bar{q}_R \not{D} q_R + y_\Phi (\bar{q}_R \Phi q_L + \bar{q}_L \Phi^\dagger q_R) , \quad (5)$$

where $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative.

The Yukawa term $\sim y_\Phi$ which couples quarks to the chiral field Φ is an effective coupling. Including such a term is useful in constructing an effective model for the chiral transition [21]. We only write this term in order to contrast the difference between the possible effective couplings between quarks and the tetraquark fields in Eqs. (18) and (29).

We note that it is possible for chiral symmetry to be broken not by a quark antiquark operator in the $\bar{\mathbf{3}} \times \mathbf{3}$ representation, but by a four quark operator in the $\mathbf{8} \times \mathbf{8}$ representation [22, 23]. These four quark operators differ from the tetraquark operators which we consider. In QCD, though, there are general arguments against this possibility [23], and certainly no indication from numerical simulations on the lattice that this occurs [24–26].

III. TWO FLAVORS

The most attractive channel for the scattering of two quarks is antisymmetric in both flavor and color [1]. For two flavors, a diquark in this channel is then an anti-triplet in color and an isosinglet in flavor,

$$\chi_L^A = \epsilon^{ABC} \epsilon^{ab} (q_L^{aB})^T \mathcal{C}^{-1} q_L^{bC} , \quad (6)$$

where \mathcal{C} is the charge conjugation matrix [5]. In a basis where $\gamma_5 = (\mathbf{1}_2, -\mathbf{1}_2)$ is diagonal, $\mathcal{C} = \text{diag}(-\sigma_2, \sigma_2)$. The transpose of the quark field and the charge conjugation matrix \mathcal{C} are necessary to form a Lorentz scalar. This combination is naturally related to the diquark condensates for color superconductivity [18–20].

To obtain a spin zero field we combine left handed diquark with a right handed diquark to form

$$\zeta = (\chi_R^A)^* \chi_L^A . \quad (7)$$

The tetraquark field ζ is a color singlet and complex valued. It is invariant under $SU(2)_L \times SU(2)_R$, but transforms under axial $U(1)_A$ as

$$\zeta \rightarrow e^{-2i\alpha} \zeta , \quad (8)$$

so that ζ has axial $U(1)_A$ charge = -2 .

Unlike for Φ , we can also form tetraquark fields from diquarks of the *same* chirality:

$$\zeta_L = (\chi_L^A)^* \chi_L^A , \quad \zeta_R = (\chi_R^A)^* \chi_R^A . \quad (9)$$

Both ζ_L and ζ_R are real valued and singlets under all flavor transformations. Thus while they can be constructed, there is no reason to expect that they should significantly affect the dynamics in any interesting way. In particular, they appear through terms which are linear in themselves, and have an expectation value at any temperature.

We thus turn to constructing an effective Lagrangian which couples the usual chiral field Φ and the tetraquark field ζ under an exact chiral symmetry of $SU(2)_L \times SU(2)_R$.

We assume that in counting mass dimensions, all scalar fields have mass dimension one, as holds for a fundamental scalar in four spacetime dimensions. Since the quarks have mass dimension $3/2$, this is different from their nominal mass dimension, which is three for Φ , and six for ζ . This is, however, a standard assumption in constructing effective models, and is

certainly justified by the renormalization group near a transition of second order. We then categorize all terms up to quartic order in Φ and ζ .

While in the chiral limit the $SU(2)_L \times SU(2)_R$ symmetry is exact, the axial $U(1)_A$ symmetry is only valid classically, and is spontaneously broken quantum-mechanically by topologically nontrivial configurations such as instantons [27, 28]. There still persists a discrete axial symmetry of $Z(2)_A$. The simplest operator which is invariant under $SU(2)_L \times SU(2)_R$, but not $U(1)_A$, is the determinant of Φ . For two flavors, under axial $U(1)_A$ this operator has axial charge $= +2$,

$$\det\Phi \rightarrow e^{2i\alpha} \det\Phi . \quad (10)$$

This is invariant if $\alpha = 0$ or π , which is the residual symmetry of axial $Z(2)_A$.

Consequently, any couplings which invariant under $Z(2)_A$ but not $U(1)_A$ are nonzero in vacuum and for a range of temperature. Eventually, at high temperature the breaking of axial $U(1)_A$ is only due to instantons. This is suppressed by a high power of temperature [27], so that axial $U(1)_A$ is effectively restored as the temperature $T \rightarrow \infty$. This is supported by numerical simulations on the lattice [29].

To help categorize the possible terms in effective potentials it helps to start with those which persist at high temperature, where axial $U(1)_A$ is an approximate symmetry. The $U(1)_A$ invariant terms that only involve Φ are

$$\mathcal{V}_\Phi^\infty = m_\Phi^2 \text{tr}(\Phi^\dagger\Phi) + \lambda_{\Phi 1} \text{tr}(\Phi^\dagger\Phi)^2 + \lambda_{\Phi 2} (\text{tr}\Phi^\dagger\Phi)^2 . \quad (11)$$

These terms are standard in linear sigma models. For two flavors, $|\det\Phi|^2$ is also a quartic term, but because $\Phi^\dagger\Phi$ is a Hermitian matrix, this can be expressed as a sum of the two terms above, $|\det\Phi|^2 = \det\Phi^\dagger\Phi = ((\text{tr}\Phi^\dagger\Phi)^2 - \text{tr}(\Phi^\dagger\Phi)^2)/2$.

There are two $U(1)_A$ invariant terms which only involve ζ ,

$$\mathcal{V}_\zeta^\infty = m_\zeta^2 |\zeta|^2 + \lambda_\zeta (|\zeta|^2)^2 . \quad (12)$$

Lastly, there are two $U(1)_A$ invariant terms coupling Φ and ζ ,

$$\mathcal{V}_{\zeta\Phi}^\infty = +\kappa_\infty (\zeta \det\Phi + \text{c.c.}) + \lambda_{\zeta\phi 1} |\zeta|^2 \text{tr}(\Phi^\dagger\Phi) . \quad (13)$$

The last term is unremarkable, as both $|\zeta|^2$ and $\text{tr}(\Phi^\dagger\Phi)$ are each separately invariant under $U(1)_A$. The first term, however, is novel: it is a trilinear coupling between one ζ field, with

axial charge = -2 , and two Φ 's, with charge = $+1$. Adding the complex conjugate (c.c.) assures the total term is real. There is an analogous term for three flavors, Eq. (24).

We then move on to categorize the full set of terms which contribute at finite temperature, where the $U(1)_A$ symmetry is reduced to $Z(2)_A$. There are three terms involving only Φ :

$$\mathcal{V}_\Phi^A = \kappa_\Phi (\det\Phi + \text{c.c.}) + \lambda_{\Phi 3} (\det\Phi + \text{c.c.}) \text{tr}(\Phi^\dagger\Phi) + \lambda_{\Phi 4} (\det\Phi + \text{c.c.})^2 . \quad (14)$$

The first is a mass term makes the η meson heavy, and so splits the $U(1)_A$ symmetry in the spectrum [28]. The other two are couplings of quartic order. Since Φ itself is not a Hermitian matrix, $\det\Phi$ does not reduce to traces of Φ , and these are new, independent couplings. (This can be checked by taking the elements of Φ to be only off-diagonal and complex.)

At zero temperature, the tetraquark field ζ is a $Z(2)_A$ singlet, and so there is no symmetry relating the real and imaginary parts of ζ . The real part of ζ , ζ_r , is even under parity, while the imaginary part, ζ_i , is odd. We start with the terms for ζ_r . As it is a $Z(2)_A$ singlet and parity even, the couplings of ζ_r with itself involves *arbitrary* powers:

$$\mathcal{V}_{\zeta_r}^A = h_r \zeta_r + m_r^2 \zeta_r^2 + \kappa_r \zeta_r^3 + \lambda_r \zeta_r^4 . \quad (15)$$

Assuming that the underlying theory, such as QCD, does not spontaneously break parity, then only even powers of the imaginary part ζ_i can appear. Otherwise, arbitrary combinations of ζ_r and ζ_i^2 enter:

$$\mathcal{V}_{\zeta_i}^A = +m_i^2 \zeta_i^2 + \kappa_i \zeta_r \zeta_i^2 + \lambda_{i1} \zeta_i^4 + \lambda_{i2} \zeta_r^2 \zeta_i^2 . \quad (16)$$

However, ζ_i does not play a significant role in the chiral phase transition, and so we neglect it henceforth.

That leaves couplings between ζ_r and Φ ,

$$\mathcal{V}_{\zeta\Phi}^A = \kappa_{\zeta\Phi} \zeta_r \text{tr}(\Phi^\dagger\Phi) + \lambda_{\zeta\Phi 2} \zeta_r^2 (\det\Phi + \text{c.c.}) . \quad (17)$$

The trilinear coupling between ζ_r and $\text{tr}(\Phi^\dagger\Phi)$ was noticed first by Giacosa [8].

These effective Lagrangians can be used to analyze the effect of the tetraquark field ζ on the chiral phase transition. At zero temperature we assume that the chiral symmetry is spontaneously broken, with $\langle\Phi\rangle \neq 0$. Further, since there is no reason why h_r in Eq. (15) should vanish, we also assume that $\langle\zeta_r\rangle \neq 0$ at $T = 0$.

As the temperature changes all of the $U(1)_A$ invariant couplings in Eqs. (11), (12), and (13) are nonzero at any T . In contrast, the $Z(2)_A$ invariant couplings in Eqs. (14), (15), (16), and (17) vanish as $T \rightarrow \infty$.

In particular, while ζ_r and ζ_i are not related at zero temperature, as $T \rightarrow \infty$, we should have the (approximate) restoration of axial $U(1)_A$ symmetry. This implies that ζ_r and ζ_i are degenerate, with $\langle \zeta_r \rangle \rightarrow 0$ as $T \rightarrow \infty$.

A chiral phase transition occurs when the expectation value of Φ vanishes. We first review the standard picture in the absence of the tetraquark field ζ . If axial $U(1)_A$ is badly broken at T_χ , then the chiral symmetry is $SU(2)_L \times SU(2)_R$. Assuming that quartic couplings are positive at T_χ , when m_Φ^2 vanishes there is a second order phase transition in the universality class of $O(4)$ symmetry. If $Z(2)_A$ is approximately $U(1)_A$ by T_χ then the chiral transition could be induced to be first order through fluctuations [13]. Another possibility is that $SU(2)_L \times SU(2)_R \times U(1)_A = O(4) \times O(2)$ has an infrared stable fixed point in a new universality class [14]. For analyses in effective models, see Ref. [15].

Including the tetraquark field ζ does not appear to significantly affect the chiral phase transition. For two flavors, *all* of the mixing terms between ζ and Φ , Eqs. (13) and (17), are quadratic in Φ . Consequently, the mixing between Φ and ζ is $\sim \langle \Phi \rangle$. If the chiral transition is of second order, at T_χ this mixing vanishes, and only Φ is a critical field. Both ζ_r and ζ_i are massive fields which mix with Φ due to cubic terms.

If the chiral transition for two flavors is of first order, then of course both $\langle \Phi \rangle$ and $\langle \zeta_r \rangle$ have a discontinuity at T_χ .

Given the generality of the potentials, it is possible that there is a phase transition associated with ζ_r , independent of that for Φ . Even if h_r vanishes at one given temperature, due to the cubic terms in ζ_r , $\langle \zeta_r \rangle$ should still be nonzero. As noted by Mukherjee and Huang [17], this does not exclude the possibility of a first order transition at which $\langle \zeta_r \rangle$ jumps discontinuously from one value to another. While possible, however, there is compelling reason why such a first order transition in ζ_r should occur.

We briefly discuss the mass spectrum of the model. As a complex valued field, Φ has components with $J^P = 0^+$ and 0^- . The 0^+ is composed of an isosinglet, the σ , and an isotriplet, analogous to the \vec{a}_0 . For the 0^- part we have an isosinglet η and an isotriplet of pions, $\vec{\pi}$. In addition, ζ contains two isosinglet fields, $\zeta_{r,i}$ with $J^P = 0^\pm$.

At zero temperature, $\langle \Phi \rangle \neq 0$ generates a massless pion and η , and a massive σ and \vec{a}_0 .

Terms which are only invariant under $Z(2)_A$ make the η massive, pushing the mass of the σ down [28]. With the tetraquark field, all that happens is that the ζ_r field mixes with the σ , as does the ζ_i field with the η ; the isotriplet states are unaffected.

As noted above, the mixing between Φ and ζ is $\sim \langle \Phi \rangle$, and so vanishes in the chirally symmetric phase. At very high temperatures where $U(1)_A$ symmetry is approximately valid, the Φ multiplet is (nearly) degenerate, as are ζ_r and ζ_i . There is no reason why the masses of Φ and ζ should be related to one another, although the two fields couple through Eq. (13).

We conclude by noting that because the tetraquark field is a singlet under $SU(2)_L \times SU(2)_R$, there is no Yukawa coupling analogous that between Φ and the quark fields, Eq. (5). There is, however, a $U(1)_A$ invariant coupling,

$$y_{\zeta 2} \left((\chi_L^A)^* \zeta \chi_R^A + (\chi_R^A)^* \zeta^* \chi_L^A \right) , \quad (18)$$

using $\chi_{L,R}^A$ from Eq. (6). As each $\chi_{L,R}$ is a diquark operator, this is a coupling between ζ and *four* quarks, so the coupling $y_{\zeta 2} \sim 1/\text{mass}^3$. A coupling with such a large, negative mass dimension is much less important than those given above, which have either positive or vanishing mass dimension.

IV. THREE FLAVORS

A. Tetraquarks with opposite chirality

For three flavors the diquark field is

$$\chi_L^{aA} = \epsilon^{abc} \epsilon^{ABC} (q_L^{bB})^T \mathcal{C}^{-1} q_L^{cC} . \quad (19)$$

Because of the anti-symmetric tensor, χ_L transforms as an anti-triplet, $\bar{\mathbf{3}}$, in both color and flavor. The diquark fields χ_L and χ_R can be combined into a color singlet, tetraquark field,

$$\zeta^{ab} = (\chi_R^{aA})^* \chi_L^{bA} . \quad (20)$$

Unlike two flavors, ζ transforms nontrivially under the $SU(3)_L \times SU(3)_R$ chiral symmetry:

$$\zeta \rightarrow e^{-2i\alpha} U_R \zeta U_L^\dagger . \quad (21)$$

Note that while we define $\Phi \sim \bar{q}_L q_R$, as a left-right field Eq. (3), we choose to define $\zeta \sim \chi_R^\dagger \chi_L \sim \bar{q}_R \bar{q}_R q_L q_L$ as right-left. We do this so that both ζ and Φ in the same way under $SU(3)_L \times SU(3)_R$, as $\bar{\mathbf{3}} \times \mathbf{3}$. Because of this difference, they have opposite signs under the axial $U(1)_A$ symmetry: Φ^{ab} has axial charge +1, while ζ^{ab} has charge -2.

As for two flavors, we first categorize the interactions which are $U(1)_A$ invariant. Those involving just Φ are

$$\mathcal{V}_\Phi^\infty = m_\Phi^2 \text{tr} (\Phi^\dagger \Phi) + \lambda_{\Phi 1} \text{tr} (\Phi^\dagger \Phi)^2 + \lambda_{\Phi 2} (\text{tr} (\Phi^\dagger \Phi))^2 , \quad (22)$$

and similarly for ζ ,

$$\mathcal{V}_\zeta^\infty = m_\zeta^2 \text{tr} (\zeta^\dagger \zeta) + \lambda_{\zeta 1} \text{tr} (\zeta^\dagger \zeta)^2 + \lambda_{\zeta 2} (\text{tr} (\zeta^\dagger \zeta))^2 . \quad (23)$$

Even under the assumption of $U(1)_A$ symmetry, there are numerous couplings between ζ and Φ . The most interesting is a trilinear coupling between ζ and Φ ,

$$\mathcal{V}_{\Phi,3}^\infty = \kappa_\infty \epsilon^{abc} \epsilon^{a'b'c'} \left(\zeta^{aa'} \Phi^{bb'} \Phi^{cc'} + \text{c.c.} \right) . \quad (24)$$

This term ties left handed indices with left handed, and right with right, and so is invariant under $SU(3)_L \times SU(3)_R$. (Note that both Φ^{ab} and ζ^{ab} are defined so that the first index is for $SU(3)_R$, and the second for $SU(3)_L$.) This is invariant under the axial $U(1)_A$ symmetry because there is one ζ with charge -2 and two Φ 's with charge +1. This coupling is analogous to that for two flavors, $\sim \kappa_\infty \zeta \det \Phi$ in Eq. (13).

There are four quartic couplings which are invariant under $U(1)_A$ and mix ζ and Φ :

$$\begin{aligned} \mathcal{V}_{\Phi,4}^\infty &= \lambda_{\zeta\Phi 1} \text{tr} (\zeta^\dagger \zeta \Phi^\dagger \Phi) + \lambda_{\zeta\Phi 2} \text{tr} (\zeta^\dagger \Phi \Phi^\dagger \zeta) \\ &+ \lambda_{\zeta\Phi 3} \text{tr} (\zeta^\dagger \zeta) \text{tr} (\Phi^\dagger \Phi) + \lambda_{\zeta\Phi 4} \text{tr} (\zeta^\dagger \Phi) \text{tr} (\Phi^\dagger \zeta) . \end{aligned} \quad (25)$$

We next turn to terms which are invariant only under $Z(3)_A$ and not $U(1)_A$. The most important was noted first by Black, Fariborz, and Schechter [2]. This is a quadratic term, which *directly* mixes ζ and Φ ,

$$\mathcal{V}_{\Phi,2}^A = m_{\zeta\Phi}^2 \text{tr} (\zeta^\dagger \Phi + \Phi^\dagger \zeta) . \quad (26)$$

This has axial charge ± 3 and so is $Z(3)_A$ invariant. The existence of this mixing term is an immediate consequence of the fact that ζ and Φ transform in the same representation of the chiral symmetry group.

There are three cubic terms which are $Z(3)_A$ invariant,

$$\mathcal{V}_{\zeta\Phi,3}^A = \kappa_\Phi (\det\Phi + \text{c.c.}) + \kappa_\zeta (\det\zeta + \text{c.c.}) + \kappa_{\zeta\Phi} \epsilon^{abc} \epsilon^{a'b'c'} \left(\zeta^{aa'} \zeta^{bb'} \Phi^{cc'} + \text{c.c.} \right). \quad (27)$$

The last term is clearly similar to that in Eq. (24), except that it involves two ζ 's and one Φ , with axial charge ∓ 3 .

There are six quartic terms which are $Z(3)_A$ invariant,

$$\begin{aligned} \mathcal{V}_{\zeta\Phi,4}^A = & \lambda_{\zeta\Phi 5} (\text{tr}(\zeta^\dagger \zeta \zeta^\dagger \Phi) + \text{c.c.}) + \lambda_{\zeta\Phi 6} (\text{tr}(\zeta^\dagger \Phi)^2 + \text{c.c.}) + \lambda_{\zeta\Phi 7} (\text{tr}(\zeta^\dagger \Phi \Phi^\dagger \Phi) + \text{c.c.}) \\ & + \lambda_{\zeta\Phi 8} (\text{tr}(\zeta^\dagger \zeta) \text{tr}(\zeta^\dagger \Phi) + \text{c.c.}) + \lambda_{\zeta\Phi 9} \left((\text{tr}(\zeta^\dagger \Phi))^2 + \text{c.c.} \right) \\ & + \lambda_{\zeta\Phi 10} (\text{tr}(\zeta^\dagger \Phi) \text{tr}(\Phi^\dagger \Phi) + \text{c.c.}). \end{aligned} \quad (28)$$

These terms agree with Fariborz, Jora, and Schechter, Appendix A in Refs. [3] and [4].

As discussed before for two flavors, we assume that all couplings which are invariant under $Z(3)_A$ but not $U(1)_A$ are large at zero temperature, but negligible at high temperature. We do not assume that they are small at the chiral phase transition.

At zero temperature we expect that the chiral symmetry is broken by a nonzero expectation value for Φ , $\langle \Phi \rangle \neq 0$. Since we take the chiral symmetry to be exact, $\langle \Phi \rangle$ is proportional to the unit matrix. Because of the mixing term in Eq. (26), an expectation value for Φ automatically induces one for ζ , with $\langle \zeta \rangle \neq 0$.

At high temperature we expect the chiral symmetry is restored, so $\langle \Phi \rangle = \langle \zeta \rangle = 0$. Further, because the direct mixing between Φ and ζ , Eq. (26), which is only invariant under $Z(3)_A$, at high temperature the masses of ζ and Φ do not mix. The fields do interact through $U(1)_A$ invariant couplings such as Eqs. (24) and (25).

The interesting question is how chiral symmetry is restored. This depends upon the details of the effective Lagrangian. For example, assume that m_ζ^2 is *very* large and positive at zero temperature. Then an expectation value of ζ is induced only by its mixing with Φ : the phase transition is driven by the interactions of Φ with itself, and ζ plays a tangential role.

Since both Φ and ζ lie in the same representation of $SU(3)_L \times SU(3)_R$, the converse is also possible: if m_Φ^2 is large and positive at zero temperature, then chiral symmetry breaking and restoration is driven by the tetraquark field, ζ .

We suggest that it is possible that both Φ and ζ play important roles in the breaking of chiral symmetry at zero temperature, and its restoration at T_χ .

If so, then it is very possible that there are *two* chiral phase transitions, at temperatures $T_{\tilde{\chi}}$ and T_{χ} , where $T_{\tilde{\chi}} < T_{\chi}$. Because of the cubic terms in ζ and Φ , Eqs. (24) and (27), both transitions are presumably of first order. As the temperature increases from zero, there is first a phase transition at $T_{\tilde{\chi}}$, where both $\langle\Phi\rangle$ and $\langle\zeta\rangle$ jump discontinuously. Because of their mixing, both condensates remain nonzero above but close to $T_{\tilde{\chi}}$. As the temperature continues to increase, they jump again at T_{χ} , and vanish for $T > T_{\chi}$. Thus T_{χ} is properly termed the temperature for the restoration of chiral symmetry. Nevertheless, the transition at $T_{\tilde{\chi}}$ is also a chiral phase transition, since both expectation values jump there. It is simply not a transition above which the chiral symmetry is restored.

In terms of the effective Lagrangian, there is a wide range of parameters in which there are two chiral phase transitions. The most obvious is if the mass squared of both Φ and ζ are negative at zero temperature. Then given the bounty of cubic terms, it is extremely *unnatural* for there to be only one phase transition.

What we are suggesting is actually rather elementary. If both the usual chiral field Φ and the tetraquark field ζ matter at zero temperature, as suggested by hadronic phenomenology, then because they lie in the *same* representation of $SU(3)_L \times SU(3)_R$, it is very plausible that each chiral field drives a phase transition.

As shown by our discussion of two flavors, our conclusion is special to three flavors. As we discuss in Sec. (VIII), even for four flavors the relevant fields may differ, and be hexaquarks instead of tetraquarks.

The importance of tetraquarks for the chiral phase transition is also special to being close to the chiral limit. For physical values of the quark masses, numerical simulations on the lattice find only one chiral phase transition [24, 25]; for recent reviews, see [26]. As we argue in the next section, the tetraquark field becomes more important as the quarks become lighter.

We conclude this section by noting that unlike the case of two flavors, that the tetraquark field can couple directly to quarks through a Yukawa interaction similar to that for Φ in Eq. (5),

$$y_{\zeta 3} \left(\bar{q}_R \zeta q_L + \bar{q}_L \zeta^\dagger q_R \right) . \quad (29)$$

However, this coupling has axial charge ∓ 3 , and so is invariant under $Z(3)_A$, but not $U(1)_A$. Thus $y_{\zeta 3}$ vanishes as $T \rightarrow \infty$.

B. Tetraquarks with the same chirality

Analogous to the case of two flavors, Eq. (9), it is also possible to combine two diquark fields with the same chirality:

$$\zeta_L^{ab} = \chi_L^{aA} (\chi_L^{bA})^* \quad , \quad \zeta_R^{ab} = \chi_R^{aA} (\chi_R^{bA})^* . \quad (30)$$

By their definition these fields are Hermitian, $\zeta_L^\dagger = \zeta_L$ and $\zeta_R^\dagger = \zeta_R$. They transform as an adjoint field under the associated flavor group, with axial charge zero:

$$\zeta_L \rightarrow U_L \zeta_L U_L^\dagger \quad , \quad \zeta_R \rightarrow U_R \zeta_R U_R^\dagger . \quad (31)$$

For the left handed fields, their self interaction include

$$\mathcal{V}_{\zeta_L} = h_{\zeta_L} \text{tr}(\zeta_L) + m_{\zeta_L}^2 \text{tr}(\zeta_L^2) + \kappa_{\zeta_L} \text{tr}(\zeta_L^3) + \lambda_{\zeta_L} \text{tr}(\zeta_L^4) ; \quad (32)$$

and similarly for ζ_R .

The important point is that because they are Hermitian fields, and carry zero charge under axial $U(1)_A$, then a term linear in either the trace of ζ_L or ζ_R is allowed at any temperature. Thus we expect that each develops a nonzero expectation value. This is true at any temperature: even in the chirally symmetric phase, if $\langle \zeta_L \rangle$ and $\langle \zeta_R \rangle$ are each proportional to the unit matrix, then $SU(3)_L$ and $SU(3)_R$ symmetries remain unbroken by these expectation values. We assume this remains valid at any temperature.

Couplings of the left and right handed tetraquark fields with Φ include

$$\mathcal{V}_{\zeta_L \Phi} = \kappa_{\zeta_L \Phi} \text{tr}(\zeta_L \Phi^\dagger \Phi) + \kappa_{\zeta_R \Phi} \text{tr}(\zeta_R \Phi \Phi^\dagger) + \lambda_{\zeta_L \Phi} \text{tr}(\zeta_L \Phi^\dagger \zeta_R \Phi) ; \quad (33)$$

plus other terms. Invariance under parity requires $\kappa_{\zeta_L \Phi} = \kappa_{\zeta_R \Phi}$. There are, of course, also couplings with the left-right tetraquark field ζ as well as with Φ . If both ζ_L and ζ_R develop expectation values which are proportional to the unit matrix, however, then all of these terms reduce to couplings just between Φ and ζ , as written down previously. For example, all of the terms in Eq. (33) reduce just to $\text{tr}(\Phi^\dagger \Phi)$. Consequently, we do not expect that whatever happens with ζ_L and ζ_R to materially affect the phase transitions in Φ and ζ . As for two flavors [17], there can be first order transitions associated with either field at any temperature, but there seems to be no compelling dynamical reason for such transitions.

V. TOY MODEL

In this section we discuss a simple model which illustrates how two chiral phase transitions can arise for three massless flavors.

A. Single chiral field

We first review the chiral phase transition for a single chiral field, Φ . Besides establishing notation, it helps to illustrate the range of possible values. We start with the Lagrangian of Eqs. (22) and (27),

$$\mathcal{V}_\Phi(\Phi) = m^2 \text{tr}(\Phi^\dagger \Phi) + \kappa (\det \Phi + \text{c.c.}) + \lambda \text{tr}(\Phi^\dagger \Phi)^2. \quad (34)$$

To avoid notational clutter, we drop the subscript Φ , taking $m_\Phi^2 = m^2$, $\kappa_\Phi = \kappa$, and $\lambda_{\Phi 1} = \lambda$. We also drop the coupling $\sim \lambda_{\Phi 2}(\text{tr}(\Phi^\dagger \Phi))^2$.

In the chiral limit we take the expectation value of Φ to be diagonal,

$$\langle \Phi^{ab} \rangle = \phi \delta^{ab}. \quad (35)$$

For this value,

$$\mathcal{V}_\Phi(\phi) = 3m^2 \phi^2 - 2\kappa \phi^3 + 3\lambda \phi^4. \quad (36)$$

The equation of motion for ϕ is

$$\frac{\partial \mathcal{V}_\Phi(\phi)}{\partial \phi} = 6\phi(m^2 - \kappa \phi + 2\lambda \phi^2). \quad (37)$$

In the chiral limit, there are only four distinct masses. The fields with $J^P = 0^-$ are a degenerate octet, composed of the pions, kaons, and the η , and a singlet η' . Those with $J^P = 0^+$ are a degenerate octet of the a_0 's, K^* 's, and an f_0 meson, and a singlet σ meson.

These four masses can be read off from Eqs. (68), (71), (77), and (81) of Ref. [21],

$$\begin{aligned} m_\pi^2 &= m^2 - \kappa \phi + 2\lambda \phi^2, \\ m_{\eta'}^2 &= m^2 + 2\kappa \phi + 2\lambda \phi^2, \\ m_{a_0}^2 &= m^2 + \kappa \phi + 6\lambda \phi^2, \\ m_\sigma^2 &= m^2 - 2\kappa \phi + 6\lambda \phi^2. \end{aligned} \quad (38)$$

The pion mass squared is proportional to the equation of motion, Eq. (37), and so $m_\pi^2 = 0$, as necessary for a Goldstone boson.

It is illuminating to rewrite the couplings in terms of these masses. The equation of motion can be written as

$$m_{\eta'}^2 - m_\pi^2 = m_{a_0}^2 - m_\sigma^2 . \quad (39)$$

Although here the pion mass vanishes, this relation remains valid even when $m_\pi^2 \neq 0$, Eq. (91) of Ref. [21]. Using this relation, we can express all three parameters in terms of ϕ and two masses,

$$\begin{aligned} m^2 &= \frac{1}{6} (m_{\eta'}^2 - 3 m_\sigma^2) , \\ \kappa \phi &= \frac{1}{3} m_{\eta'}^2 , \\ \lambda \phi^2 &= \frac{1}{12} (m_{\eta'}^2 + 3 m_\sigma^2) . \end{aligned} \quad (40)$$

As expected, the η' is massive because of the determinantal coupling $\sim \kappa$. Notice that the expectation value ϕ is not fixed by these relations; usually that is determined by the value of the pion decay constant.

Usually, in mean field theory one assumes that only the mass parameter m^2 is a function of temperature, and takes κ and λ to be constant. At high temperature $m^2(T) \sim \lambda T^2$, but the dependence is more complicated at small temperature. We do not need to know this dependence to determine the masses and couplings at the chiral phase transition, T_χ .

The solutions to the equation of motion are

$$\phi(T) = \frac{\kappa}{4\lambda} \left(1 \pm \sqrt{1 - \frac{8\lambda}{\kappa^2} m^2(T)} \right) , \quad (41)$$

where in the broken phase the minimum corresponds to the $+$ sign. The transition occurs when the free energy, which is minus the potential, is equal to that in the symmetric phase. Since $\mathcal{V}_\Phi(0) = 0$, this occurs when

$$\mathcal{V}_\Phi(\phi(T_\chi)) = 0 \quad \Rightarrow \quad \frac{8\lambda}{\kappa^2} m^2(T_\chi) = + \frac{1}{9} . \quad (42)$$

Just below the transition temperature,

$$T = T_\chi^- : \quad m_{\eta'} = \sqrt{\frac{\kappa^2}{\lambda}} , \quad m_\sigma = \frac{1}{3} m_{\eta'} , \quad m_{a_0} = \sqrt{\frac{10}{9}} m_{\eta'} . \quad (43)$$

For $T > T_\chi^+$, all masses, including those for the pion, $= m(T)$. Assuming that $m^2(T)$ is monotonically increasing with temperature, in order to have a phase transition we need that $m^2(T_\chi) > m^2(0)$.

The precise mass spectrum for QCD with three massless flavors is not known at present. The relations in Eq. (40) show that we can always treat two masses as free parameters. As an example, consider

$$T = 0 : \quad m_{\eta'} = 960 \quad , \quad m_{\sigma} = 600 \quad , \quad m_{a_0} = 1130 \quad , \quad m^2 = -(162)^2 \quad , \quad (44)$$

where all masses are in MeV. In this we assume that the η' and the σ mesons have the values given above, and stress that they are only meant as *suggestive*. The mass of the a_0 meson follows from Eq. (39), and m^2 from Eq. (40). In QCD the masses of the η' and the a_0 are very close, but the above value for m_{a_0} isn't so unreasonable in the chiral limit.

Using these values we can then compute the corresponding quantities at the chiral transition temperature:

$$T = T_{\chi} : \quad m_{\eta'} = 752 \quad , \quad m_{\sigma} = 251 \quad , \quad m_{a_0} = 835 \quad , \quad m^2 = +(89)^2 \quad . \quad (45)$$

While all masses decrease with increasing temperature, at T_{χ} those for the η' and the a_0 are still $\sim 75 - 80\%$ of their values at $T = 0$, while that for the σ meson is only $\sim 40\%$. That doesn't tell us the value of T_{χ} , since that depends upon the details of the temperature dependence of $m^2(T)$.

B. Mirror model at zero temperature

We now construct the simplest possible model which illustrates how two chiral phase transitions arise in the chiral limit. We assume that the potential for the tetraquark field ζ is given by

$$\mathcal{V}_{\zeta}(\zeta) = m^2 \text{tr} (\zeta^{\dagger} \zeta) + \kappa (\det \zeta + \text{c.c.}) + \lambda \text{tr} (\zeta^{\dagger} \zeta)^2 \quad . \quad (46)$$

We term this the “mirror” model, since we choose all parameters to be identical to those for Φ , Eq. (34): comparing to Eq. (23), we take $m_{\zeta}^2 = m^2$, $\kappa_{\zeta} = \kappa$, and $\lambda_{\zeta} = \lambda$.

The only term that we include which mixes ζ and Φ is the $Z(3)_A$ invariant term in Eq. (26). It is not difficult to see that including just this term greatly alters the mass spectrum. Taking an expectation value for ζ which is diagonal, the mixing term is

$$\langle \zeta^{ab} \rangle = \zeta \delta^{ab} \quad : \quad \mathcal{V}_{\text{mix}} = 3 \tilde{m}^2 \zeta \phi \quad , \quad (47)$$

where again to simplify the notation we take $\tilde{m}^2 = m_{\zeta\Phi}^2$. Henceforth in this section, by ζ we denote not the matrix, but the scalar expectation value of the diagonal component thereof.

From Eq. (37), the equations of motion become

$$\begin{aligned}\frac{\partial}{\partial\phi}(\mathcal{V}_\Phi + \mathcal{V}_{\text{mix}}) &= 6(\tilde{m}^2\zeta + m^2\phi - \kappa\phi^2 + 2\lambda\phi^3) , \\ \frac{\partial}{\partial\zeta}(\mathcal{V}_\zeta + \mathcal{V}_{\text{mix}}) &= 6(\tilde{m}^2\phi + m^2\zeta - \kappa\zeta^2 + 2\lambda\zeta^3) .\end{aligned}\quad (48)$$

For each field, the mixing term acts like a background field proportional to the other field: in the equation of motion for ϕ , there is a term $\sim \tilde{m}^2\zeta$, and *vice versa*.

We first compute the mass spectrum at zero temperature. At $T = 0$, where $m_\zeta^2 = m_\Phi^2$ the expectation values ζ and ϕ are equal. The exact value is not of relevance for our purposes.

The Φ fields contains two octets, which we term the π and a_0 , and two singlets, the η' and the σ . There are similar fields for the ζ , which we denote as the $\tilde{\pi}$, \tilde{a}_0 , $\tilde{\eta}'$, and $\tilde{\sigma}$. The mixing between these fields which is induced by Eq. (26) is particularly simple: the π mixes only with the $\tilde{\pi}$, the η' only with the $\tilde{\eta}'$, and so on. Finding the mass eigenstates then requires diagonalizing four 2×2 matrices. For the pions, using the equation of motion in Eq. (48), when $\zeta = \phi$ the mass matrix between the π and the $\tilde{\pi}$ is

$$\mathcal{M}_{\pi\tilde{\pi}}^2 = \tilde{m}^2 \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} . \quad (49)$$

The eigenvalues of this matrix are

$$\pi, \tilde{\pi} : \quad 0 \quad , \quad -2\tilde{m}^2 . \quad (50)$$

For the mass squared of the massive “pion” to be positive requires that \tilde{m}^2 is negative. This is unremarkable, as the mass squared for both the ζ and Φ are also negative. Since the expectation values of ζ and ϕ are equal, after diagonalization each mass eigenstate is a linear combination of the original fields, in equal proportion.

Since we are in the chiral limit, there is one massless and one massive octet. There are nine Goldstone bosons when $SU(3)_L \times SU(3)_R \times U(1)_A$ symmetry breaks to $SU(3)$. The quantum breaking of $U(1)_A$ to $Z(3)_A$ makes the η' massive and reduces this to eight. When the ζ and Φ are decoupled each has eight Goldstone bosons. Coupling them makes one of the octets massive, leaving one massless octet required by Goldstone’s theorem.

The mass squared for the remaining fields are

$$\begin{aligned}\eta', \tilde{\eta}' : \quad & 3\kappa\phi \quad , \quad 3\kappa\phi - 2\tilde{m}^2 ; \\ a_0, \tilde{a}_0 : \quad & m^2 + \kappa\phi + 6\lambda\phi^2 \pm \tilde{m}^2 ; \\ \sigma, \tilde{\sigma} : \quad & m^2 - 2\kappa\phi + 6\lambda\phi^2 \pm \tilde{m}^2 .\end{aligned}\quad (51)$$

These are naturally related to the masses in the absence of mixing, given in Eq. (38). Notice, however, that the expectation value of ϕ is the solution of Eq. (48), and not the solution of Eq. (37).

The masses of the a_0 and \tilde{a}_0 , and the σ and $\tilde{\sigma}$, are elementary, just the mass splitting induced by off-diagonal elements $\sim \tilde{m}^2$ in the mass matrix. The η' and $\tilde{\eta}'$ differ from these because they are Goldstone bosons when $\kappa = 0$. For all of these mesons, the mass eigenstates are linear superpositions of the original fields, although not equally. The exact mixing is easy to work out.

The masses in Eq. (51) obey the relation

$$m_{\eta'}^2 + m_{\tilde{\eta}'}^2 - m_{\pi}^2 - m_{\tilde{\pi}}^2 = m_{a_0}^2 + m_{\tilde{a}_0}^2 - m_{\sigma}^2 - m_{\tilde{\sigma}}^2. \quad (52)$$

One can show that given the potentials of Eqs. (34) and (46), this relation remains valid even if we do not assume that the parameters are related as a mirror model.

We have not checked that this relation remains valid for arbitrary potentials, but even so it illustrates a more general point. The relation for two fields in Eq. (52) is very similar to that for one field in Eq. (39). For one field, however, there is a puzzle. As expected, the anomaly term $\sim \kappa \det \Phi$ splits the singlet η' from octet π , making the η' heavy. However, it also pushes the mass of the singlet σ *down* relative to the octet a_0 . Now of course to compare to QCD we need to include the effect of quark masses, especially that the strange quark is heavier than the up and down. Even so, it is peculiar that the *singlet* σ is lighter than the octet a_0 for the $J^P = 0^+$ field.

With two fields, however, there is no problem, as the only relation is between the *sum* of the masses squared. Thus the anomaly pushes the sum of the mass squared of the η' and the $\tilde{\eta}'$ up relative to that for π and the $\tilde{\pi}$. Conversely, the anomaly pushes the masses of both the σ *and* the $\tilde{\sigma}$ down relative to the a_0 and the \tilde{a}_0 . Since the states from the ζ can be significantly heavier than the usual states, it is much easier satisfying this constraint. See also our discussion in Sec. (VI A).

C. Mirror model at nonzero temperature

All of these masses in the previous section can only be valid at zero temperature, where by fiat we imposed the condition that $m_{\zeta}^2(0) = m_{\Phi}^2(0)$. At nonzero temperature, because

the states are linear combinations of the original fields, with mixing due to \tilde{m}^2 , then

$$m_\zeta^2(T) \neq m_\Phi^2(T) \quad (53)$$

at *any* nonzero temperature.

This is obvious from effective models. If one computes the thermal fluctuations from the ζ and Φ fields, then just because the masses of the two multiplets differ, so will the effective masses for ζ and Φ . In other words, we can tune the masses to be equal at a given temperature, such as zero, but we cannot impose this naturally at another temperature.

For example, in the limit of high temperature, if we neglect mesonic fluctuations then the dominant contribution to the thermal masses are given by quark loops. For the Φ field this is $\sim y_\Phi T^2$, where y_Φ is the Yukawa coupling of the quarks to the Φ , Eq. (5), while for ζ it is $\sim y_{\zeta 3} T^2$, Eq. (29). There is no symmetry which relates the two Yukawa couplings, and so $y_\Phi \neq y_{\zeta 3}$. Indeed, since the coupling $\sim y_{\zeta 3}$ respects the axial $Z(3)_A$ symmetry but not $U(1)_A$, $y_{\zeta 3}$ vanishes as $T \rightarrow \infty$, while y_Φ is nonzero. Thus the two masses differ as $T \rightarrow \infty$, Eq. (53).

As an example, we assume that we fix the expectation values at zero temperature to agree with the value of the pion decay constant in QCD, which is $\phi(0) = 93/2$, Eq. (93) of Ref. [21]. From Eqs. (40) and (44),

$$\phi(0) = \zeta(0) = 46. \quad , \quad \kappa = 6680. \quad , \quad \lambda = 79. \quad . \quad (54)$$

These values are similar to those from a fit to QCD, Eqs. (95) and (96) of Ref. [21]. Notice that in a linear sigma model that the couplings κ and λ are so large because the pion decay constant is much smaller than the masses of the η' and the σ .

We consider three cases to illustrate the range of possibilities. In the first case, we take

$$m_\phi^2(T) = 3T^2 + m^2(0) \quad , \quad m_\zeta^2(T) = 5T^2 + m^2(0) \quad , \quad m_{\zeta\Phi}^2 = -(100)^2 \quad . \quad (55)$$

We stress that the temperature dependence is meant only to be illustrative. At low temperatures massless pions give a contribution $\sim T^2$, but the other contributions from massive fields are Boltzmann. In this instance, the order parameters behave as in Fig. (1). There are two first order phase transitions: at $T_{\tilde{\chi}}$, both ζ and ϕ jump from one nonzero value to another. At T_χ , both jump from nonzero values to zero.

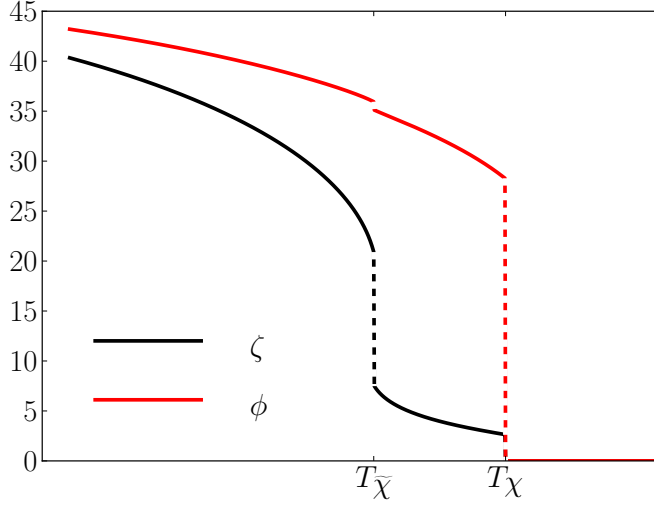


FIG. 1. The temperature dependence of the order parameters, ζ and ϕ , in the mirror model for the parameters of Eq. (55). There are two chiral transitions of first order, at $T_{\tilde{\chi}}$ and then T_{χ} .

In the second case, we only change the mixing mass $m_{\zeta\Phi}^2$,

$$m_{\phi}^2(T) = 3T^2 + m^2(0) \quad , \quad m_{\zeta}^2(T) = 5T^2 + m^2(0) \quad , \quad m_{\zeta\Phi}^2 = -(120)^2 \quad . \quad (56)$$

The order parameters behave as in Fig. (2). Because the mixing mass $m_{\zeta\Phi}^2$ is larger, the mixing term acts as a larger background field. This smooths out the would be transition at $T_{\tilde{\chi}}$ from first order to crossover. There is then a single chiral phase transition at T_{χ} .

These examples are only meant to illustrate what is possible, and should only be taken as such. Nevertheless, it clearly is possible to obtain a second chiral phase transition from the presence of the tetraquark condensate.

So far we have only considered the chiral limit. To understand the broader implications for the phase diagram of QCD we need to consider how nonzero quark masses affect a tetraquark condensate and the phase diagram.

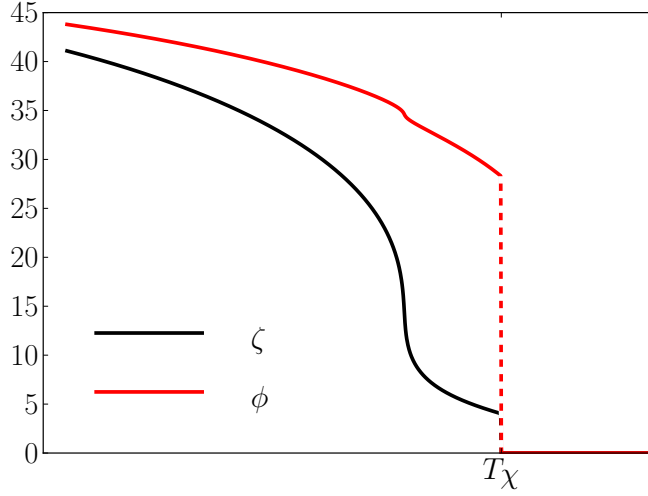


FIG. 2. The temperature dependence of the order parameters, ζ and ϕ , in the mirror model, with the parameters of Eq. (56). There is one chiral phase transition of first order at T_χ .

VI. MASSIVE QUARKS

A. Mass terms for tetraquarks

To describe QCD it is necessary to include terms for the explicit breaking of chiral symmetry. Let the current quark masses be

$$\mathcal{M} = \text{diag}(m_u, m_d, m_s), \quad (57)$$

where m_u , m_d , and m_s are the masses for the up, down, and strange quarks.

Since m_u and m_d are much less than other scales in QCD, we take the isospin symmetric limit with $m_u = m_d$. In a sigma model, the breaking of chiral symmetry is represented including a background field proportional to the mass matrix,

$$\mathcal{V}_\Phi^1 = -\text{tr} (H_\Phi (\Phi^\dagger + \Phi)) , \quad (58)$$

with

$$H_\Phi = (h_u, h_u, h_s) . \quad (59)$$

If chiral symmetry is approximately valid we expect that the ratio of the h 's is proportional to that for the current quark masses, $h_u/h_s = m_u/m_s$. However, the overall constant is given by the details of the fit to the sigma model.

For small quark masses, it suffices to include only terms linear in H and Φ . A complete catalog of all possible terms is given in Appendix A of Refs. [3] and [6].

To understand the leading mass term for the tetraquark field, imagine computing it explicitly in perturbation theory. This is of course a terrible approximation, but it should suffice to get the leading powers of the quark mass right. For the usual Φ field, its expectation value is proportional to a quark loop, $\sim \text{tr}(1/(\not{D} + m_{quark}))$. For small masses this trace is proportional to m_{quark} , so $\langle \Phi^{ii} \rangle \sim m_{quark}$, which is given by taking $H^{ii} \sim m_{quark}$, Eq. (59).

The tetraquark field, however, involves the antisymmetric tensor for flavor, Eqs. (6) and (19). For example, the expectation value of the strange-strange component of the tetraquark field, ζ^{ss} , involves the product of an up quark loop times a down quark loop. For small quark masses each is proportional to the mass, so $\langle \zeta^{ss} \rangle \sim m_u m_d$. The other components follow similarly. Hence for the tetraquark field, the leading term which breaks the chiral symmetry is proportional to the *square* of the quark masses, and is

$$\mathcal{M}_\zeta = \text{diag}(m_d m_s, m_u m_s, m_u m_d) . \quad (60)$$

Assuming $SU(2)$ isospin symmetry,

$$\mathcal{M}_\zeta \approx m_u \text{diag}(m_s, m_s, m_u) . \quad (61)$$

Thus to the linear sigma model we add

$$\mathcal{V}_\zeta^1 = -\text{tr}(H_\zeta(\zeta^\dagger + \zeta)) . \quad (62)$$

Assuming $SU(2)$ isospin symmetry,

$$H_\zeta = h(h_s, h_s, h_u) . \quad (63)$$

There is no reason for the background field for ζ to be identical to that for Φ , and so while we expect that in H_ζ we have $h \sim h_u$, we should take h as an independent constant to be fit by hadronic phenomenology.

If we can neglect mixing, then the mass term of Eq. (63) immediately gives us insight into why tetraquarks are so appealing in QCD. As we discussed at the end of Sec. (VB) following Eq. (48), a sigma model with a single field Φ gives a light σ which has a large strange component. For the tetraquark field, however, the mass term for the strange-strange component of ζ is proportional to the product of the *light* quark masses, $\sim m_u m_d$. That is,

a mass term such as Eq. (61) naturally gives an “inverted” mass ordering which appears to be present in QCD for the lightest 0^+ multiplet [1–10].

This of course neglects mixing between the Φ and ζ fields, in particular through the direct mixing in Eq. (26). This term does induce an expectation value $\langle\zeta\rangle \sim m_{\zeta\Phi}^2\langle\Phi\rangle \sim m_{\zeta\Phi}^2 H_\Phi$. As we stressed in Sec. (V B), however, *all* fields are linear combinations of Φ and ζ . Different choices for the parameters of the model gives different ratios of mixtures.

It is still meaningful to stress that the leading term in quark masses for the tetraquark field is that of Eq. (63). For example, in the limit of high temperature the mixing term $\sim m_{\zeta\Phi}^2$ is very small, and the breaking of the chiral symmetry from explicit quark masses is much smaller for ζ than for Φ .

B. Phase diagram for three light flavors

In this section we make discuss the implications for the phase diagram in moving away from the chiral limit. As seen in the discussion of the mirror model in Sec. (V C), in the chiral limit it is possible to obtain two chiral phase transitions, at $T_{\tilde{\chi}}$ and T_χ .

A useful way of plotting the phase diagram versus the quark masses is in the two dimensional plane of the light quark mass, taking $m_u = m_d$, versus the strange quark mass m_s .

When all quark masses are large, there is a region of first order phase transitions which are dominated by that for deconfinement. In a matrix model [30] the critical line which borders this region of first order deconfining phase transitions is determined by the color $Z(3)$ field generated by heavy quarks. Whatever bound states the heavy quarks form — whether of two, four, or however many quarks — seems unlikely to affect the position of the critical line for deconfinement.

Thus we concentrate on the region of small quark masses. *If* there are two chiral phase transitions in the chiral limit, $m_u = m_d = m_s$, then it is natural that this persists for a nonzero width in the plane of m_u and m_s . We illustrate this in the “Columbia” phase diagram of Fig. (3). Thus region II denotes where there are two chiral phase transitions for first order, ending in the dotted line. In region I, there is one chiral transition of first order, ending in the solid line. QCD lies in C, the crossover region.

Both the dotted and solid lines are regions where there is a critical line. That there is a

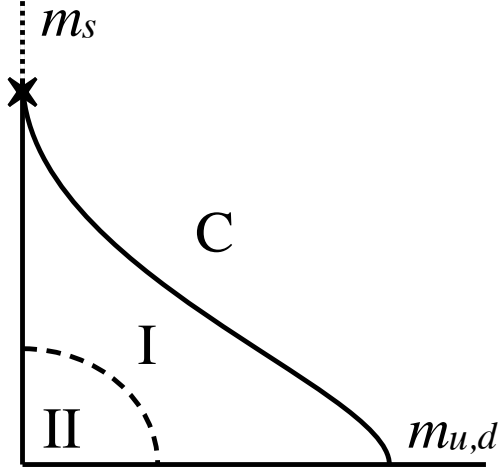


FIG. 3. The phase diagram for three light flavors of quarks, in the plane of $m_u = m_d$ versus m_s . In region II there are two first order chiral phase transitions; in region I, one transition; in region C, there is only crossover. There are critical lines separating regions II and I, and I and C. QCD lies in the crossover region.

critical line in going from two to one chiral phase transition can be guessed from the behavior of the order parameters in Fig. (2). Along the dotted critical line, at a temperature $T_{\tilde{\chi}}$ there is a linear combination of the Φ and ζ fields which is critical. This transition is separate from the first order chiral transition at T_{χ} .

The most interesting part of Fig. (3) is the left most axis, where

$$m_u = m_d = 0 \quad , \quad m_s \neq 0 \quad \Rightarrow \quad H_{\Phi} = (0, 0, h_s) \quad , \quad H_{\zeta} = (0, 0, 0) \quad . \quad (64)$$

Notice that H_{ζ} vanishes because it is proportional to m_u , Eq. (63).

Consider first the strange-strange component of the tetraquark field, ζ^{ss} . Because of mixing with Φ , it develops a nonzero expectation value. Then ζ^{ss} acts exactly like the tetraquark field for two flavors. For instance, the $U(1)_A$ invariant trilinear coupling $\sim \kappa_{\infty}$ for three flavors, Eq. (24), reduces directly to the $U(1)_A$ invariant trilinear coupling for two flavors $\sim \kappa_{\infty}$ in Eq. (13). In agreement with our arguments about two flavors in Sec. (III), we do not expect that the strange-strange component of the tetraquark field significantly affects the chiral transition when $m_u = m_d = 0$, Eq. (64).

When $m_u = m_d = 0$ there is one subtlety which is worth noting. Assume that the effects of the anomaly are large, so that we can assume that there is only a $Z(2)_A$ symmetry, and not $U(1)_A$. For the quark masses as in Eq. (64), from the $SU(3)_L \times SU(3)_R$ fields ζ and Φ

we can obviously extract two $O(4)$ fields, $\vec{\zeta}$ and $\vec{\phi}$. The effective Lagrangian for these two $O(4)$ fields is

$$\begin{aligned} \mathcal{V}_{m_u=m_d=0} = & m_\phi^2 \vec{\phi}^2 + m_\zeta^2 \vec{\zeta}^2 + m_{\zeta\phi}^2 \vec{\zeta} \cdot \vec{\phi} \\ & + \lambda_\phi (\vec{\phi}^2)^2 + \lambda_\zeta (\vec{\zeta}^2)^2 + \lambda_{\zeta\phi 1} \vec{\zeta}^2 \vec{\phi}^2 + \lambda_{\zeta\phi 2} (\vec{\zeta} \cdot \vec{\phi})^2. \end{aligned} \quad (65)$$

The couplings above are obviously related to those we denoted previously. Clearly, there is no trilinear coupling between $\vec{\zeta}$ and $\vec{\phi}$ which is $O(4)$ invariant.

It is not difficult to convince oneself that having two fields doesn't alter the standard picture. For small m_s , in regions II and I, there are either two or one chiral phase transition. Approaching the boundary of region I from below, there is a tricritical point at m_s^{tri} , denoted by a cross in Fig. (3). For $m_s > m_s^{tri}$, there is a line of second order phase transitions, in the universality class of $O(4)$. Unless there is a conspiracy in the masses for $\vec{\zeta}$ and $\vec{\phi}$, though, having two fields doesn't alter the critical behavior in the least: it simply means that some linear combination of $\vec{\zeta}$ and $\vec{\phi}$ is the relevant critical field. The shape of the critical line between regions I and C as $m_s \rightarrow 0$ is dictated by the tricritical behavior. There is no such curvature for the critical line, separating regions I and II, because the transition remains of second order as one moves along the critical line when $m_s \rightarrow 0$.

Of course this assumes that there is a region II with two chiral transitions of first order. This question can only be settled definitively by numerical simulations on the lattice. Since QCD only finds a crossover, such simulations need to be done for very light quarks, which is most challenging. Nevertheless, if the lattice does find two chiral phase transitions for light quarks, would be strong if indirect evidence for the effects of tetraquarks in QCD.

VII. PHASE DIAGRAM IN T AND μ

In QCD, at nonzero temperature but zero quark chemical potential, numerical simulations on the lattice indicate that there is no true phase transition, but only a crossover for a single chiral transition at a temperature of $T_\chi \sim 155$ MeV [24–26]. Thus any second chiral transition associated with the tetraquark field, at $T_{\tilde{\chi}} < T_\chi$, is almost certainly a crossover.

Even so, at nonzero temperature and quark chemical potential, there is naturally a relation between the crossover line for tetraquark field and the transition line for color superconductivity. A tetraquark field is important because of diquark pairing, with the most

attractive channel for quark-quark scattering being antisymmetric in both flavor and color, as a type of generalized Breit interaction [1]. Thus it is hardly surprising that in considering the scattering of two quarks at the edge of the Fermi sea at nonzero density, that color superconductivity occurs in the corresponding, most attractive channel.

The analogy is deeper. Consider the diquark operators for two flavors, χ_L^A in Eq. (6), and three flavors, χ_L^{aA} in Eq. (19). These are almost identical to the operators which condense when color superconductivity occurs [18–20]. That is, the tetraquark field is *directly* the gauge invariant square of the diquark operators, Eqs. (7) and (20). Of course the tetraquark field must be a color singlet, since it appears in the confined phase at zero temperature and chemical potential; there is no evidence for a color superconducting phase in vacuum.

There are differences between the condensation of a tetraquark field in vacuum and color superconductivity. Color superconductivity is dominated by the scattering of quarks at opposite edges of the Fermi surface, between two quarks with momenta $+\vec{p}_F$ and $-\vec{p}_F$. For a tetraquark condensate, the entire tetraquark field carries zero momentum, but each diquark operators carries equal and opposite momenta. Further, the color-flavor locking which occurs for three flavors and three colors [18, 19] has no analogy for the tetraquark condensate.

Even so, as one moves out in quark chemical potential, then it is reasonable to speculate that a crossover line for the tetraquark condensate connects smoothly with that for color superconductivity.

We illustrate this in Fig. (4), as a cartoon of the possible phase diagram. In particular, we do not indicate whether the transitions are crossover, or true phase transitions, of either first or second order. The chiral crossover line at $\mu_B = 0$ may end in a critical endpoint, and then turn first order [31]. The transition line for color superconductivity probably includes a segment which is a line of second order phase transitions [20]. Further, it is not evident how the tetraquark/color superconducting line is related to that for hadronic superfluidity through a confined but dense quarkyonic phase [32].

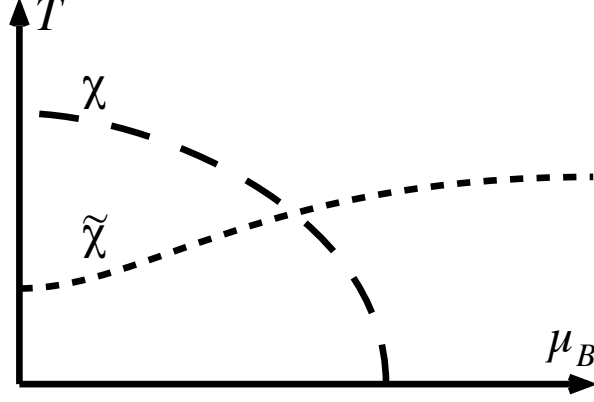


FIG. 4. A conjectured phase diagram in temperature T and baryon chemical potential μ_B . The χ line is that for the chiral transition. The $\tilde{\chi}$ line is for a second transition related to the presence of the tetraquark condensate, which may connect smoothly to the transition line for color superconductivity.

VIII. FOUR FLAVORS

We conclude with elementary comment about operators for four flavors and three colors. The diquark operator which directly generalizes those for two and three flavors is

$$\chi_L^{(ab)A} = \epsilon^{abcd} \epsilon^{ABC} (q_L^{cB})^T \mathcal{C}^{-1} q_L^{dC} . \quad (66)$$

This operator is an antisymmetric two-index tensor in the $SU(4)_L$ flavor group, a $\bar{\mathbf{6}}$. As before we then combine a left handed diquark field with a right handed diquark to form a color singlet tetraquark field,

$$\zeta^{(ab),(cd)} = \left(\chi_R^{(ab)A} \right)^\dagger \chi_L^{(cd)A} , \quad (67)$$

where $\zeta^{(ab),(cd)} = -\zeta^{(ba),(cd)}$, *etc.*, so ζ lies in the $\bar{\mathbf{6}} \times \mathbf{6}$ representation of $SU(4)_L \times SU(4)_R$.

Since the tetraquark field ζ lies in a higher representation of the chiral symmetry group than Φ , there is no direct mixing between them. There is a cubic coupling,

$$i \left(\Phi^{aa'} \right)^* \left(\zeta^{(ab),(a'b')} - \left(\zeta^{(ab),(a'b')} \right)^* \right) \Phi^{bb'} , \quad (68)$$

where the overall factor of i follows from the antisymmetry of ζ . This coupling is invariant under $Z(4)_A$ but not $U(1)_A$. There are many quartic couplings which can be written down, including $U(1)_A$ invariant terms such as

$$\text{tr} (\Phi^\dagger \Phi) \text{tr} (\zeta^\dagger \zeta) . \quad (69)$$

There are also quartic couplings which are invariant only under $Z(4)_A$, such as

$$\Phi^{aa'} \zeta^{(ab),(a'b')} \zeta^{(bc),(b'c')} \Phi^{cc'} + \text{c.c.} . \quad (70)$$

Consequently, when Φ develops a vacuum expectation value it affects ζ . Even so, there are no couplings present which would indicate that the presence of ζ materially affects either the pattern of symmetry breaking, or its restoration, in any significant way.

This is a more general phenomenon. Starting with QCD, from the quark fields it is possible to construct a ladder of operators related to chiral symmetry breaking. The simplest is Φ , which transforms as the fundamental representation in $SU(N_f)_L \times SU(N_f)_R$, where N_f is the number of flavors. In addition, there are four quark operators, six quark operators, and so on. Categorizing them according to the representations of the chiral symmetry group, typically they are either singlets, such as $\text{tr}(\Phi^\dagger \Phi)$, or transform according to higher representations. Singlet fields are like the tetraquark for two flavors in Sec. (III), which don't dramatically affect things. Similarly, fields in higher representations do couple to fields in lower representations, but again it is unnatural for them to have any dramatic effect. This is just because fields in high representations have more indices, and so as illustrated in Eqs. (68), (69), and (70), need more Φ 's to absorb all of them.

The one exception is if there is another field which transforms in the fundamental representation which directly mixes with Φ . If one abandons the prejudice of only using diquarks fields, then it is possible to construct such a field for four flavors and three colors. For a large number of colors, Rossi and Veneziano suggested that junctions, which couple all colors together through an antisymmetric tensor in color, matter [33]. This suggests using junctions in both color and flavor to form a triquark operator

$$\chi_L^a = \epsilon^{abcd} \epsilon^{ABC} q_L^{bA} (q_L^{cB})^T \mathcal{C}^{-1} q_L^{dC} . \quad (71)$$

This is a color singlet, and since it is composed of three quark fields, transforms as a fermion.

We can naturally combine a left handed triquark with a right handed triquark to form a hexaquark state

$$\xi^{ab} = (\chi_R^a)^\dagger \chi_L^b , \quad (72)$$

which transforms as $\bar{\mathbf{4}} \times \mathbf{4}$ under $SU(4)_L \times SU(4)_R$.

The analysis of coupling the hexaquark field ξ to the usual chiral field Φ is very similar to that for three flavors. The axial $U(1)_A$ symmetry is reduced to $Z(4)_A$ by the anomaly, with Φ carrying axial charge = +1, and ξ , axial charge = -3.

The $U(1)_A$ invariant couplings include mass and quartic couplings between ξ and Φ , in direct analogy to Eqs. (22), (23), and (25), replacing $\zeta \rightarrow \xi$. For three flavors there is the $U(1)_A$ invariant determinental term of Eq. (24). The analogous term for four flavors is

$$\kappa_\infty \epsilon^{abcd} \epsilon^{a'b'c'd'} \xi^{aa'} \Phi^{bb'} \Phi^{cc'} \Phi^{dd'} + \text{c.c.} . \quad (73)$$

Of the couplings which are invariant under $Z(4)_A$ but not $U(1)_A$, the most important is a direct mixing term

$$m_{\xi\Phi}^2 \text{tr} (\xi^\dagger \Phi + \Phi^\dagger \xi) , \quad (74)$$

as in Eq. (26). The determinental terms $\sim \kappa_\Phi \det \Phi$ and $\sim \kappa_\xi \det \xi$ are of quartic order for four flavors. There are two determinental terms of quartic order,

$$\epsilon^{abcd} \epsilon^{a'b'c'd'} \left(\kappa_{\xi\Phi 1} \xi^{aa'} \xi^{bb'} \Phi^{cc'} \Phi^{dd'} + \kappa_{\xi\Phi 2} \xi^{aa'} \xi^{bb'} \xi^{cc'} \Phi^{dd'} \right) + \text{c.c.} . \quad (75)$$

The other quartic terms invariant under $Z(4)_A$ are those of Eq. (28), just replacing $\zeta \rightarrow \xi$.

The analysis of the chiral transition for four flavors with a hexaquark field is then closely analogous to that with a tetraquark field for three flavors. The hexaquark field ξ mixes directly with Φ , so if one field condenses both do. Similarly, the restoration of chiral symmetry involves both fields. For four flavors the determinental terms are of quartic instead of cubic order, and so do not automatically generate first order transitions.

However, it is also possible that chiral transitions are driven first order by fluctuations, where coupling constants flow from positive to negative values. This occurs to leading order in an ϵ -expansion about $4 - \epsilon$ dimensions when $N_f > \sqrt{2}$ [13]. It is not clear if this remains true in three dimensions: for two flavors, there is evidence that a new critical point develops for $SU(2)_L \times SU(2)_R \times U(1)_A = O(4) \times O(2)$ [14, 15]. If so, it is possible that such a new critical point persists up to four flavors. We note that even with two fields in the fundamental representation, only one linear combination of the two contributes to the putative critical behavior at T_χ . If the transition is of first order, it is also possible that there are having two chiral fields in the fundamental representation produces two chiral transitions, as for three flavors.

It is interesting to speculate what the relevant effective fields are for the chiral transition when the number of colors, N_c , is greater than three. It has been suggested that tetraquarks persist in the usual large N_c limit, where N_f is held fixed as $N_c \rightarrow \infty$ [34]. On the other hand, our analysis suggests that the relevant limit might be more general: taking $N_f = N_c \rightarrow \infty$,

instead of tetraquarks we would use junctions in both flavor and color to form $2(N_c-1)$ quark states which transform in the fundamental representation of the chiral symmetry group.

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- [1] R. J. Jaffe, Physical Review D **15**, 267 (1977); Physical Review D **15**, 267 (1977); R. L. Jaffe, arXiv:hep-ph/0001123 (2000), arXiv: hep-ph/0001123; R. L. Jaffe and F. Wilczek, Physical Review Letters **91** (2003), 10.1103/PhysRevLett.91.232003, arXiv: hep-ph/0307341; R. L. Jaffe, Physics Reports **409**, 1 (2005), arXiv: hep-ph/0409065.
 - [2] D. Black, A. H. Fariborz, and J. Schechter, Physical Review D **61** (2000), 10.1103/PhysRevD.61.074001.
 - [3] A. H. Fariborz, R. Jora, and J. Schechter, Physical Review D **72** (2005), 10.1103/PhysRevD.72.034001, arXiv: hep-ph/0506170.
 - [4] A. H. Fariborz, R. Jora, and J. Schechter, Physical Review D **77** (2008), 10.1103/PhysRevD.77.034006, arXiv: 0707.0843.
 - [5] A. H. Fariborz, R. Jora, and J. Schechter, Physical Review D **77** (2008), 10.1103/PhysRevD.77.094004.
 - [6] D. Black, A. H. Fariborz, F. Sannino, and J. Schechter, Physical Review D **59** (1999), 10.1103/PhysRevD.59.074026, arXiv: hep-ph/9808415; A. H. Fariborz, R. Jora, and J. Schechter, Physical Review D **76** (2007), 10.1103/PhysRevD.76.014011, arXiv: hep-ph/0612200; Physical Review D **79** (2009), 10.1103/PhysRevD.79.074014, arXiv: 0902.2825; A. H. Fariborz, arXiv:1109.2630 [hep-ph] (2011), arXiv: 1109.2630; A. H. Fariborz, R. Jora, J. Schechter, and M. N. Shahid, Physical Review D **84** (2011), 10.1103/PhysRevD.84.113004; A. H. Fariborz, J. Schechter, S. Zarepour, and M. Zebarjad, Physical Review D **90** (2014), 10.1103/PhysRevD.90.033009, arXiv: 1407.3870; A. H. Fariborz, A. Azizi, and A. Asrar, Physical Review D **91** (2015), 10.1103/PhysRevD.91.073013.

- [7] N. A. Tornqvist, arXiv:hep-ph/0204215 (2002), arXiv: hep-ph/0204215; F. E. Close and N. A. Tornqvist, Journal of Physics G: Nuclear and Particle Physics **28**, R249 (2002), arXiv: hep-ph/0204205; C. Amsler and N. A. Tornqvist, Phys.Rept. **389**, 61 (2004); M. Napsuciale and S. Rodriguez, Physical Review D **70** (2004), 10.1103/PhysRevD.70.094043, arXiv: hep-ph/0407037; Physics Letters B **603**, 195 (2004), arXiv: hep-ph/0403072; L. Maiani, F. Piccinini, A. D. Polosa, and V. Riquer, Physical Review Letters **93** (2004), 10.1103/PhysRevLett.93.212002; J. R. PelEz, Modern Physics Letters A **19**, 2879 (2004); M. R. Pennington, Progress of Theoretical Physics Supplement **168**, 143 (2007), arXiv: hep-ph/0703256; G. 't Hooft, G. Isidori, L. Maiani, A. Polosa, and V. Riquer, Physics Letters B **662**, 424 (2008); J. R. Pelez, M. R. Pennington, J. Ruiz de Elvira, and D. J. Wilson, Physical Review D **84** (2011), 10.1103/PhysRevD.84.096006; W. Heupel, G. Eichmann, and C. S. Fischer, Physics Letters B **718**, 545 (2012), arXiv: 1206.5129; T. K. Mukherjee, M. Huang, and Q.-S. Yan, Physical Review D **86** (2012), 10.1103/PhysRevD.86.114022; H.-X. Chen, E.-L. Cui, W. Chen, T. G. Steele, X. Liu, and S.-L. Zhu, Physical Review D **91** (2015), 10.1103/PhysRevD.91.094022; G. Eichmann, C. S. Fischer, and W. Heupel, Physics Letters B **753**, 282 (2016).
- [8] F. Giacosa, Physical Review D **75** (2007), 10.1103/PhysRevD.75.054007.
- [9] D. Parganlija, F. Giacosa, and D. H. Rischke, Physical Review D **82** (2010), 10.1103/PhysRevD.82.054024; F. Giacosa, The European Physical Journal C **65**, 449 (2010), arXiv: 0907.3519; S. Janowski, D. Parganlija, F. Giacosa, and D. H. Rischke, Physical Review D **84** (2011), 10.1103/PhysRevD.84.054007, arXiv: 1103.3238; D. Parganlija, P. Kovacs, G. Wolf, F. Giacosa, and D. H. Rischke, Physical Review D **87** (2013), 10.1103/PhysRevD.87.014011, arXiv: 1208.0585; T. Wolkanowski and F. Giacosa, Acta Physica Polonica B Proceedings Supplement **7**, 469 (2014), arXiv: 1404.5758; S. Janowski, F. Giacosa, and D. H. Rischke, Physical Review D **90** (2014), 10.1103/PhysRevD.90.114005, arXiv: 1408.4921; Z. Ghalenovi, F. Giacosa, and D. H. Rischke, arXiv:1507.03345 [hep-ph] (2015), arXiv: 1507.03345; F. Giacosa, Acta Physica Polonica B **47**, 7 (2016), arXiv: 1511.04605.
- [10] K. Olive, Chinese Physics C **38**, 090001 (2014); J. R. Pelaez, arXiv:1510.00653 [hep-ph, physics:nucl-th] (2015), arXiv: 1510.00653.
- [11] M. Karliner, EPJ Web of Conferences **71**, 00065 (2014), arXiv: 1401.4058; S. Stone, arXiv:1509.04051 [hep-ex, physics:hep-ph, physics:nucl-ex, physics:nucl-th] (2015), arXiv:

1509.04051.

- [12] H. Goldberg, Phys.Lett. **B131**, 133 (1983).
- [13] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984).
- [14] A. Pelissetto and E. Vicari, Physical Review D **88** (2013), 10.1103/PhysRevD.88.105018; Y. Nakayama and T. Ohtsuki, Physical Review D **89** (2014), 10.1103/PhysRevD.89.126009, arXiv: 1404.0489; Physical Review D **91** (2015), 10.1103/PhysRevD.91.021901, arXiv: 1407.6195.
- [15] E. Meggiolaro and A. Mord, Physical Review D **88** (2013), 10.1103/PhysRevD.88.096010, arXiv: 1309.4598; G. Fejos, Physical Review D **90** (2014), 10.1103/PhysRevD.90.096011, arXiv: 1409.3695; Physical Review D **92** (2015), 10.1103/PhysRevD.92.036011, arXiv: 1506.07399; T. Sato and N. Yamada, Physical Review D **91** (2015), 10.1103/PhysRevD.91.034025, arXiv: 1412.8026; J. Eser, M. Grah, and D. H. Rischke, Physical Review D **92** (2015), 10.1103/PhysRevD.92.096008, arXiv: 1508.06928.
- [16] A. Heinz, S. Struber, F. Giacosa, and D. H. Rischke, Physical Review D **79** (2009), 10.1103/PhysRevD.79.037502, arXiv: 0805.1134; S. Gallas, F. Giacosa, and G. Pagliara, Nuclear Physics A **872**, 13 (2011), arXiv: 1105.5003; A. Heinz, F. Giacosa, and D. H. Rischke, Nuclear Physics A **933**, 34 (2015), arXiv: 1312.3244.
- [17] T. K. Mukherjee and M. Huang, Physical Review D **89** (2014), 10.1103/PhysRevD.89.076002.
- [18] M. Alford, K. Rajagopal, and F. Wilczek, Nuclear Physics B **537**, 443 (1999), arXiv: hep-ph/9804403; T. Schaefer and F. Wilczek, Physical Review Letters **82**, 3956 (1999), arXiv: hep-ph/9811473; R. D. Pisarski and D. H. Rischke, Physical Review D **60** (1999), 10.1103/PhysRevD.60.094013, arXiv: nucl-th/9903023; arXiv:nucl-th/9907094 (1999), arXiv: nucl-th/9907094; Physical Review D **61** (2000), 10.1103/PhysRevD.61.051501, arXiv: nucl-th/9907041.
- [19] K. Rajagopal and F. Wilczek, arXiv:hep-ph/0011333 (2000), arXiv: hep-ph/0011333; M. G. Alford, K. Rajagopal, T. Schaefer, and A. Schmitt, Reviews of Modern Physics **80**, 1455 (2008), arXiv: 0709.4635.
- [20] R. D. Pisarski, Phys.Rev. **C62**, 035202 (2000).
- [21] R. D. Pisarski and V. V. Skokov, arXiv:1604.00022 [hep-ph] (2016), arXiv: 1604.00022.
- [22] J. Stern, arXiv:hep-ph/9801282 (1998), arXiv: hep-ph/9801282.

- [23] I. I. Kogan, A. Kovner, and M. Shifman, *Physical Review D* **59** (1998), 10.1103/PhysRevD.59.016001.
- [24] M. Cheng, N. H. Christ, S. Datta, J. van der Heide, C. Jung, F. Karsch, O. Kaczmarek, E. Laermann, R. D. Mawhinney, C. Miao, P. Petreczky, K. Petrov, C. Schmidt, and T. Umeda, *Physical Review D* **74** (2006), 10.1103/PhysRevD.74.054507, arXiv: hep-lat/0608013; A. Bazavov, T. Bhattacharya, M. Cheng, N. H. Christ, C. DeTar, S. Ejiri, S. Gottlieb, R. Gupta, U. M. Heller, K. Huebner, C. Jung, F. Karsch, E. Laermann, L. Levkova, C. Miao, R. D. Mawhinney, P. Petreczky, C. Schmidt, R. A. Soltz, W. Soeldner, R. Sugar, D. Toussaint, and P. Vranas, *Physical Review D* **80** (2009), 10.1103/PhysRevD.80.014504, arXiv: 0903.4379; M. Cheng, N. H. Christ, S. Datta, J. van der Heide, C. Jung, F. Karsch, O. Kaczmarek, E. Laermann, R. D. Mawhinney, C. Miao, P. Petreczky, K. Petrov, C. Schmidt, W. Soeldner, and T. Umeda, *Physical Review D* **77** (2008), 10.1103/PhysRevD.77.014511, arXiv: 0710.0354; Z. Fodor and S. D. Katz, (2009); Y. Aoki, S. Borsanyi, S. Durr, Z. Fodor, S. D. Katz, S. Krieg, and K. K. Szabo, *Journal of High Energy Physics* **2009**, 088 (2009), arXiv: 0903.4155; S. Borsanyi, G. Endrodi, Z. Fodor, A. Jakovac, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabo, *Journal of High Energy Physics* **2010** (2010), 10.1007/JHEP11(2010)077, arXiv: 1007.2580; S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabo, *Journal of High Energy Physics* **2010** (2010), 10.1007/JHEP09(2010)073, arXiv: 1005.3508; M. Cheng, S. Ejiri, P. Hegde, F. Karsch, O. Kaczmarek, E. Laermann, R. D. Mawhinney, C. Miao, S. Mukherjee, P. Petreczky, C. Schmidt, and W. Soeldner, *Physical Review D* **81** (2010), 10.1103/PhysRevD.81.054504, arXiv: 0911.2215; A. Bazavov, T. Bhattacharya, M. Cheng, C. DeTar, H.-T. Ding, S. Gottlieb, R. Gupta, P. Hegde, U. M. Heller, F. Karsch, E. Laermann, L. Levkova, S. Mukherjee, P. Petreczky, C. Schmidt, R. A. Soltz, W. Soeldner, R. Sugar, D. Toussaint, W. Unger, and P. Vranas, *Physical Review D* **85** (2012), 10.1103/PhysRevD.85.054503, arXiv: 1111.1710; T. Bhattacharya, M. I. Buchoff, N. H. Christ, H.-T. Ding, R. Gupta, and others, *Phys. Rev. Lett.* **113**, 082001 (2014); A. Bazavov, T. Bhattacharya, C. DeTar, H.-T. Ding, S. Gottlieb, R. Gupta, P. Hegde, U. M. Heller, F. Karsch, E. Laermann, L. Levkova, S. Mukherjee, P. Petreczky, C. Schmidt, C. Schroeder, R. A. Soltz, W. Soeldner, R. Sugar, M. Wagner, and P. Vranas, *Physical Review D* **90** (2014), 10.1103/PhysRevD.90.094503, arXiv: 1407.6387; S. Borsanyi, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, and K. K.

- Szabo, Physics Letters B **730**, 99 (2014), arXiv: 1309.5258.
- [25] M. I. Buchoff, M. Cheng, N. H. Christ, H.-T. Ding, C. Jung, F. Karsch, Z. Lin, R. D. Mawhinney, S. Mukherjee, P. Petreczky, D. Renfrew, C. Schroeder, P. M. Vranas, and H. Yin, Physical Review D **89** (2014), 10.1103/PhysRevD.89.054514, arXiv: 1309.4149.
 - [26] H.-T. Ding, F. Karsch, and S. Mukherjee, arXiv:1504.05274 [hep-lat, physics:hep-ph, physics:nucl-ex, physics:nucl-th] (2015), arXiv: 1504.05274; C. Ratti, arXiv:1601.02367 [hep-lat] (2016), arXiv: 1601.02367.
 - [27] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Reviews of Modern Physics **53**, 43 (1981).
 - [28] G. 't Hooft, Physics Reports **142**, 357 (1986).
 - [29] S. Borsanyi, M. Dierigl, Z. Fodor, S. D. Katz, S. W. Mages, D. Nogradi, J. Redondo, A. Ringwald, and K. K. Szabo, Physics Letters B **752**, 175 (2016), arXiv: 1508.06917.
 - [30] K. Kashiwa, R. D. Pisarski, and V. V. Skokov, Phys. Rev. D **85**, 114029 (2012).
 - [31] M. Stephanov, K. Rajagopal, and E. Shuryak, Physical Review Letters **81**, 4816 (1998), arXiv: hep-ph/9806219; Physical Review D **60** (1999), 10.1103/PhysRevD.60.114028, arXiv: hep-ph/9903292.
 - [32] L. McLerran and R. D. Pisarski, Nuclear Physics A **796**, 83 (2007), arXiv: 0706.2191.
 - [33] G. Rossi and G. Veneziano, arXiv:1603.05830 [hep-ph, physics:hep-th] (2016), arXiv: 1603.05830.
 - [34] S. Weinberg, Physical Review Letters **110** (2013), 10.1103/PhysRevLett.110.261601, arXiv: 1303.0342; M. Knecht and S. Peris, Physical Review D **88** (2013), 10.1103/PhysRevD.88.036016, arXiv: 1307.1273; R. F. Lebed, Physical Review D **88** (2013), 10.1103/PhysRevD.88.057901, arXiv: 1308.2657; T. D. Cohen and R. F. Lebed, Physical Review D **89** (2014), 10.1103/PhysRevD.89.054018, arXiv: 1401.1815; T. Cohen, F. J. Llanes-Estrada, J. R. Pelaez, and J. R. de Elvira, Physical Review D **90** (2014), 10.1103/PhysRevD.90.036003, arXiv: 1405.4831; L. Maiani, A. D. Polosa, and V. Riquer, arXiv:1605.04839 [hep-ph] (2016), arXiv: 1605.04839.

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